TRINITY COLLEGE

Mathematics Specialist Unit 1&2
Test 5 2018

Calculator Free
Matrices

STUDENT'S NAME Solutions

DATE: Monday 20 August **TIME:** 21 minutes **MARKS:** 21

INSTRUCTIONS:
Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (4 marks)

Consider the following matrices:

$$A=\begin{bmatrix}2 & 0 & -1\\ 0 & -3 & 3\end{bmatrix} \qquad B=\begin{bmatrix}1 & 3 & 2\\ 7 & -3 & -1\end{bmatrix} \qquad C=\begin{bmatrix}2 & 3\end{bmatrix} \qquad D=\begin{bmatrix}-1\\ 4\end{bmatrix}$$

Determine where possible:

(a) $A-3B = \begin{bmatrix}-1 & -9 & -7\\ -21 & 6 & 6\end{bmatrix}$ [2]

(b) $DC = \begin{bmatrix}-1\\ 4\end{bmatrix}\begin{bmatrix}2 & 3\end{bmatrix}$ [2]

$$= \begin{bmatrix}-2 & -3\\ 8 & 12\end{bmatrix}$$

2. (7 marks)

Consider the following three matrices:

$$A = \begin{bmatrix} 1 & a-1 \\ 1-x & -5 \end{bmatrix} \qquad B = \begin{bmatrix} a-1 & c \\ b+2 & d+5 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -3 \\ -4 & 2c \end{bmatrix}$$

(a) Determine an expression for the value of x that will make matrix A singular. [3]

Singular $\Rightarrow$ $\det(A) = 0$

$\Rightarrow -5 \times 1 - (1-x)(a-1) = 0$

$\Rightarrow -5 - (a-1-ax+x) = 0$

$\Rightarrow ax - x = 5 + a - 1$

$\Rightarrow x = \dfrac{4+a}{a-1}$

(b) Determine the values of a, b, c and d if $B = 2C + I$, where I is the 2×2 identity matrix. [4]

$$\begin{bmatrix} a-1 & c \\ b+2 & d+5 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -8 & 4c \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow a-1 = 5 \Rightarrow a = 6$

$b+2 = -8 \Rightarrow b = -10$

$c = -6$

$d+5 = 4c+1 \Rightarrow d = 4(-6) - 4$

$= -28$

3. (5 marks)

Matrices A and B are defined as follows:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 6 \\ 7 & x^2 \end{bmatrix}$$

(a) Determine the matrix AB [2]

$$AB = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & x^2 \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 12 + 3x^2 \\ 55 & 24 + 5x^2 \end{bmatrix}$$

(b) If $AB = \begin{bmatrix} 31 & 24 \\ 55 & 44 \end{bmatrix}$, and $x < 0$, calculate the value of x. [3]

$\Rightarrow \quad 24 = 12 + 3x^2$

$\Rightarrow \quad 12 = 3x^2$

$\Rightarrow \quad x^2 = 4$

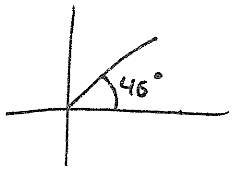
$\Rightarrow \quad x = \pm 2$

$\therefore \quad x = -2$

4. (5 marks)

Determine the Cartesian equation in exact form, of a parabola, $y = x^2$, after it has been rotated 45° anticlockwise about the origin.

Note – A Cartesian equation is expressed in terms of x and y only.



let $y = x^2$ be represented by $\begin{bmatrix} t \\ t^2 \end{bmatrix}$

$$\therefore \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} t \\ t^2 \end{bmatrix} = \begin{bmatrix} \frac{t}{\sqrt{2}} - \frac{t^2}{\sqrt{2}} \\ \frac{t}{\sqrt{2}} + \frac{t^2}{\sqrt{2}} \end{bmatrix}$$

$$\Rightarrow \quad x = \frac{t}{\sqrt{2}} - \frac{t^2}{\sqrt{2}}$$

$$y = \frac{t}{\sqrt{2}} + \frac{t^2}{\sqrt{2}}$$

Eliminating the t^2 term by adding eqns together

$$\Rightarrow \quad x + y = \frac{2t}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{\sqrt{2}}{2}(x+y) = t$$

Sub this back into on of the eqns

$$\Rightarrow \quad y = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}(x+y)\right) + \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}(x+y)\right)^2$$

$$y = \frac{1}{2}(x+y) + \frac{1}{2\sqrt{2}}(x+y)^2$$

TRINITY COLLEGE

Mathematics Specialist Unit 1&2
Test 5 2018

Calculator Assumed
Matrices

STUDENT'S NAME ______________________________

DATE: Monday 20 August **TIME:** 29 minutes **MARKS:** 29

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser
Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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5. (4 marks)

(a) If $A = \begin{bmatrix} 0 & 4 & 2 \\ 2 & 2 & 2 \\ -2 & -4 & -4 \end{bmatrix}$ determine A^2 [1]

$$A^2 = 4I \quad \Rightarrow \quad \frac{1}{4}AA = I$$

$$\Rightarrow \quad A^{-1} = \frac{1}{4}A$$

(b) Use the result from part (a) to solve the following simultaneous equations. Show your matrix equations. [3]

$$4y + 2z = -2$$
$$2x + 2y + 2z = 0$$
$$2x + 4y + 4z = -6$$

$$\Rightarrow \begin{bmatrix} 0 & 4 & 2 \\ 2 & 2 & 2 \\ -2 & -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 & 4 & 2 \\ 2 & 2 & 2 \\ -2 & -2 & -4 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

$\therefore \quad x = 3, \quad y = 2, \quad z = -5$

6. (5 marks)

Determine the matrix A, given that $A\begin{bmatrix} 6 & 5 \\ -1 & 1 \end{bmatrix} - 3A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$. Show all matrix equations.

$$\Rightarrow \quad A\left(\begin{bmatrix} 6 & 5 \\ -1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$$

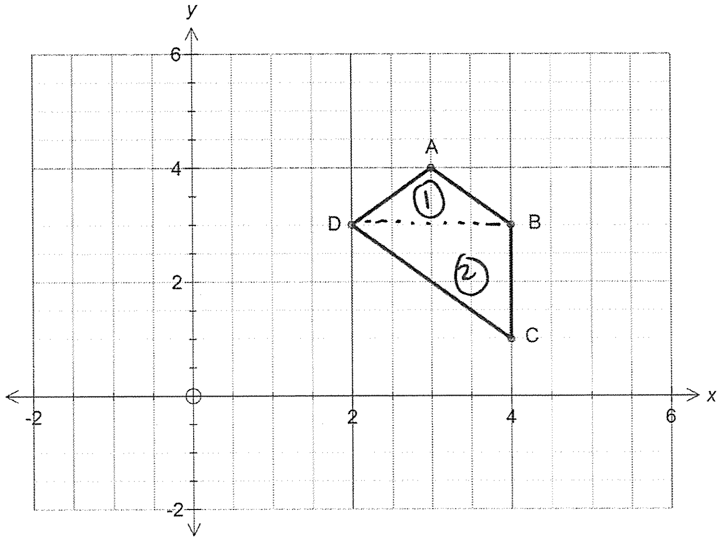
$$\Rightarrow \quad A\begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 4 & 11 \end{bmatrix}$$

7. (10 marks)

Jacob is opening his own tutoring business and decides to design a logo for his business. So far all he has is the logo drawn below.



Jacob decides to manipulate this logo by first transforming it using the matrix $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and then by using the matrix $N = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$.

(a) Describe the transformation performed by transformation matrix M. [1]

Reflection about line $y = x$

(b) Determine the image of the original points under the transformation given by M. [2]

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 & 4 & 2 \\ 4 & 3 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & 1 & 3 \\ 3 & 4 & 4 & 2 \end{bmatrix}$$

(c) Determine the single transformation that would give the same image as performing transformation M followed by transformation N. [2]

$$T = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

(d) If the original points are transformed by the matrix found in part (c), determine:

(i) the area of the original logo object. [1]

$$A = \Delta_1 + \Delta_2$$

$$= \frac{1}{2} \times 2 \times 1 + \frac{1}{2} \times 2 \times 2$$

$$= 3 \text{ units}^2$$

(ii) the area of the new logo image. [2]

$$\det(T) = 6 - 1$$

$$= 5$$

$$\therefore \text{ new area} = |5| \times 3$$

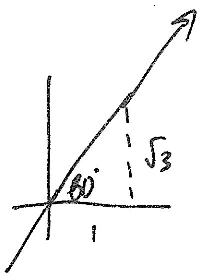
$$= 15 \text{ units}^2$$

(e) Assuming the transformation described in part (c) has taken place, determine a single matrix that would transform the new image back to the original logo. [2]

$$T^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

8. (6 marks)

An object undergoes the following sequence of transformations:



- reflection in the line $y = \sqrt{3}x$, then
- shear parallel to the x-axis with a scale factor of -2 , then
- rotation clockwise of 90°

(a) Determine a single transformation matrix to perform this sequence of transformations. [5]

$$T = \text{rot} \times \text{shear} \times \text{ref}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \sqrt{3}+\frac{1}{2} & -\frac{\sqrt{3}}{2}+1 \end{bmatrix}$$

(b) Determine which point, if any, transforms (maps) to itself. [1]

(0,0) maps to itself

9. (4 marks)

The transformation matrix, $\begin{bmatrix} 1 & a \\ 3 & 2 \end{bmatrix}$, transforms (maps) all points to a single line.

(a) Determine the value(s) of a. [2]

A line has no area $\Rightarrow \det(m) = 0$

$\Rightarrow \det \begin{bmatrix} 1 & a \\ 3 & 2 \end{bmatrix} = 0$

$\Rightarrow \quad 2 - 3a = 0$

$\Rightarrow \quad a = \frac{2}{3}$

(b) Determine the equation of the line [2]

All space maps to the line

$\therefore \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

The origin also maps to itself

$\therefore \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ an equation through $(0,0)$ and $(1,3)$ is

$y = 3x$